FUNCTORIAL CW-APPROXIMATION

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Abstract. The usual construction of a CW-approximation is functorial up to homotopy, but it is not functorial. In this note, we construct a functorial CW-approximation. Our construction takes inclusions of subspaces into inclusions of subcomplexes, and commutes with intersections of subspaces of a fixed space.

Contents

1. Introduction

A *CW-approximation* to a topological space B is a CW-complex $\widetilde{B}$ together with a weak equivalence $\widetilde{B} \to B$. The usual construction of a CW-approximation is functorial up to homotopy, but it is not functorial. In this note, we construct a functorial CW-approximation. Our construction takes inclusions of subspaces into inclusions of subcomplexes (see Theorem 2.4), and commutes with intersections of subspaces of a fixed space (see Theorem 2.5).

We construct a CW-approximation to a space using a construction that functorially factors a map $A \to B$ as $A \to \widetilde{B} \to B$ where $A \to \widetilde{B}$ is a relative CW-complex

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and $\widetilde{B} \to B$ is a weak equivalence; applying this to the map $\emptyset \to B$ produces a CW-approximation $\widetilde{B} \to B$ to B.

We actually define two such factorizations. The first is for arbitrary maps $A \to B$ (see Theorem 2.1). If A is a nonempty CW-complex, though, then the relative CW-complex $A \to \widetilde{B}$ that it produces will not, in general, be the inclusion of a subcomplex. Thus, we construct a different functorial factorization in Theorem 2.2 for maps $A \to B$ in which A is a CW-complex; in the factorization $A \to \widetilde{B} \to B$ that it produces, the relative CW-complex $A \to \widetilde{B}$ is the inclusion of a subcomplex.

We show in Theorem 2.4 that if the factorization of Theorem 2.1 is used to construct a functorial CW-approximation (by factoring the maps with domain the empty space), then this construction turns an inclusion of a subspace into an inclusion of a subcomplex, i.e., if B is a subspace of B', then $\widetilde{B}$ is a subcomplex of $\widetilde{B}'$. Thus, it defines a functorial CW-approximation for pairs, triads, etc. We also show that this operation commutes with taking intersections of subspaces of a fixed space (see Theorem 2.5).

2. The main theorems

2.1. **The first factorization.**

Theorem 2.1. *Every map $f\colon A \to B$ has a functorial factorization $A \xrightarrow{j} \widetilde{B} \xrightarrow{p} B$ such that j is a relative CW-complex and p is a weak equivalence.*

To obtain a CW-approximation $\widetilde{B} \to B$ to a space B, you apply the factorization of Theorem 2.1 to the map $\emptyset \to B$. We show in Theorem 2.4 that if B is a subspace of B' then the map $\widetilde{B} \to \widetilde{B}'$ is the inclusion of a subcomplex, and we show in Theorem 2.5 that this operation commutes with taking intersections of subspaces of a fixed space.

The outline of the proof of Theorem 2.1 follows that of the standard construction of a CW-approximation, but instead of choosing maps of spheres that represent elements of homotopy groups to be killed by attaching disks, we attach disks using all possible such maps. Thus, we attach many more cells than are required, but the result is that our construction is functorial.

This construction is a cross between the usual construction of a functorial-only-up-to-homotopy CW-approximation to a space and the small object argument used to factorize maps in model categories ([1, Prop. 10.5.16]). The standard small object argument would produce a factorization into a relative cell complex (in which the attaching maps of cells do not, in general, factor through a subspace of lower dimensional cells) followed by a map that is both a weak equivalence and a fibration; our construction produces a relative CW-complex followed by a weak equivalence. The proof of Theorem 2.1 is in Section 3.

2.2. **The second factorization.** If the space A is nonempty, then even if it is a CW-complex, the space $\widetilde{B}$ produced by Theorem 2.1 will not generally be a CW-complex, because there is no restriction on how the cells attached to construct $\widetilde{B}$ out of A meet the cells of A. Thus, we will also prove the following theorem.

Theorem 2.2. *Every map $f\colon A \to B$ such that A is a CW-complex has a functorial factorization $A \xrightarrow{j} \widetilde{B} \xrightarrow{p} B$ such that j is the inclusion of a subcomplex of a CW-complex and p is a weak equivalence, where "functorial" means that it is natural*

with respect to diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

in which $f\colon A \to A'$ *is a cellular map of CW-complexes.*

Theorem 2.2 can also be used to obtain a functorial CW-approximation to a space B by applying it to the map $\emptyset \to B$, but we show in Proposition 2.3 that this produces the same result as using Theorem 2.1.

The proof of Theorem 2.2 is in Section 4.

Proposition 2.3. *If* $\widetilde{B} \to B$ *is the CW-approximation to* B *obtained by applying the factorization of Theorem* 2.1 *to* $\emptyset \to B$ *and* $\widehat{B} \to B$ *is the CW-approximation to* B *obtained by applying the factorization of Theorem* 2.2 *to* $\emptyset \to B$*, then there is a natural isomorphism* $\widehat{B} \to \widetilde{B}$ *that makes the diagram*

$$\begin{tikzcd} \widehat{B} \arrow[dd] \arrow[dr] & \\ & B \\ \widetilde{B} \arrow[ur] & \end{tikzcd}$$

commute.

The proof of Proposition 2.3 is in Section 5.

2.3. **Relative CW-approximation.** The constructions of Theorem 2.1 and Theorem 2.2 can be used to create relative CW-approximations.

Theorem 2.4. *If* (B', B) *is a pair of spaces (i.e., if* B *is a subspace of the space* B'*) then in the commutative square*

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\tilde{f}"] \arrow[d] & \widetilde{B}' \arrow[d] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

obtained by applying the factorization of Theorem 2.1 *to the maps* $\emptyset \to B$ *and* $\emptyset \to B'$*, the map* $\tilde{f}\colon \widetilde{B} \to \widetilde{B}'$ *is an inclusion of a subcomplex.*

Thus, Theorem 2.1 creates relative CW-approximations for pairs, triads, etc. Alternatively, given a pair (B', B), one could apply Theorem 2.2 to the map $\emptyset \to B$ to obtain $\widetilde{B} \to B$ and then apply Theorem 2.2 to the composition $\widetilde{B} \to B \to B'$ to obtain $\widetilde{B}' \to B'$, and $\widetilde{B}$ would be a subcomplex of $\widetilde{B}'$. The proof of Theorem 2.4 is in Section 6.

Theorem 2.5 (CW-approximation commutes with intersections)**.** *If* X *is a space, let* $\mathrm{CW}(X)$ *denote the CW-complex obtained by applying the factorization of Theorem* 2.1 *to the map* $\emptyset \to X$*. If* X *is a space,* S *is a set, and for every element* s *of* S *we have a subspace* X_s *of* X*, then each* $\mathrm{CW}(X_s)$ *is a subcomplex of* $\mathrm{CW}(X)$*,*

and

$$\bigcap_{s\in S} \mathrm{CW}(X_s) = \mathrm{CW}\Big(\bigcap_{s\in S} X_s\Big) \ .$$

The proof of Theorem 2.5 is in Section 7.

3. The proof Theorem 2.1

We construct the factorization in Section 3.1, show that the map $\widetilde{B} \to B$ is a weak equivalence in Section 3.2, and show that the construction is functorial in Section 3.3.

3.1. **The construction.** We will construct a sequence of spaces

$$\begin{tikzcd} A = A_{-1} \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[dl] & A_1 \arrow[r] \arrow[dll] & \cdots \\ B \end{tikzcd}$$

that map to B and then let $\widetilde{B} = \mathrm{colim}_n A_n$. Each A_n for $n \geq 0$ will be constructed from A_{n-1} by attaching n-cells in such a way that the map $A_n \to B$ is n-connected (see Notation 3.1). Since spheres and disks are compact, any map from a sphere or disk to $\mathrm{colim}_n A_n$ will factor through some A_n, and so we will have $\pi_i\widetilde{B} = \mathrm{colim}_n \pi_i A_n$ for all $i \geq 0$, and the map $\widetilde{B} \to B$ will be a weak equivalence.

We begin by letting $A_{-1} = A$, and then defining

$$A_0 = A_{-1} \amalg \Big(\coprod_{D^0 \to B} D^0 \Big) \ .$$

That is, we let A_0 be the coproduct of A_{-1} with a single point for each map of a point to B; this maps to B by taking the D^0 indexed by a map $D^0 \to B$ to B by that indexing map.

To construct A_1 we construct the pushout

$$\begin{tikzcd} \displaystyle\coprod_{\mathrm{Map}(S^0,A_0)\times_{\mathrm{Map}(S^0,B)}\mathrm{Map}(D^1,B)} S^0 \arrow[rr] \arrow[dd] & & A_0 \arrow[dd] \arrow[dl, dotted] \\ & A_1 \arrow[dr, dotted] & \\ \displaystyle\coprod_{\mathrm{Map}(S^0,A_0)\times_{\mathrm{Map}(S^0,B)}\mathrm{Map}(D^1,B)} D^1 \arrow[rr] \arrow[ur, dotted] & & B \end{tikzcd}$$

where $\mathrm{Map}(S^0, A_0) \times_{\mathrm{Map}(S^0,B)} \mathrm{Map}(D^1, B)$ is the set of commutative squares

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & A_0 \arrow[d] \\ D^1 \arrow[r] & B \end{tikzcd} \ .$$

That is, for every such square we attach a 1-cell to A_0, and we use the bottom horizontal map of that square to map that attached 1-cell to B.

If $n > 1$ and we have constructed A_{n-1} along with it's map to B, we construct A_n by constructing the pushout

$$\begin{tikzcd}
\coprod_{\mathrm{Map}(S^{n-1},A_{n-1})\times_{\mathrm{Map}(S^{n-1},B)}\mathrm{Map}(D^n,B)} S^{n-1} \arrow[rr] \arrow[dd] & & A_{n-1} \arrow[dl, dotted] \arrow[dd] \\
& A_n \arrow[dr, dotted] & \\
\coprod_{\mathrm{Map}(S^{n-1},A_{n-1})\times_{\mathrm{Map}(S^{n-1},B)}\mathrm{Map}(D^n,B)} D^n \arrow[ur, dotted] \arrow[rr] & & B
\end{tikzcd}$$

where $\mathrm{Map}(S^{n-1}, A_{n-1}) \times_{\mathrm{Map}(S^{n-1},B)} \mathrm{Map}(D^n, B)$ is the set of commutative squares

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\
D^n \arrow[r] & B
\end{tikzcd} .$$

That is, for every such square we attach an n-cell to A_{n-1}, and we use the bottom horizontal map of that square to map that attached n-cell to B.

To complete the construction we let $\widetilde{B} = \mathrm{colim}_n A_n$, and the map $A \to \widetilde{B}$ is clearly a relative CW-complex. We show that the map $\widetilde{B} \to B$ is a weak equivalence in Section 3.2, and we show that the construction is natural in Section 3.3.

3.2. **The homotopy groups of the spaces in the construction.**

Notation 3.1. If $f\colon X \to Y$ is a map and $n \geq 0$, then we will say that f is *n-connected* if

- the set of path components of X maps onto the set of path components of Y, and
- for every choice of basepoint in X the induced map of homotopy groups (for $i > 0$) or sets (for $i = 0$) $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$.

Lemma 3.2. *For each $n \geq 0$ the map $A_n \to B$ is n-connected.*

Proof. We will show inductively on n that the map $A_n \to B$ is n-connected.

The space A_0 was constructed to map onto B, and so the map $A_0 \to B$ is 0-connected

The space A_1 was constructed by attaching 1-cells to A_0 that connected any pair of points in A_0 whose images were in the same path component of B; thus, the set of path components of A_1 maps isomorphically to the set of path components of B. In addition, a loop was wedged at every point of A_0 for every loop in B at the image of that point; thus, for every basepoint of A_1, the fundamental group of A_1 maps epimorphically onto the fundamental group of B. Thus, the map $A_1 \to B$ is 1-connected.

Suppose now that $n > 1$ and that the map $A_{n-1} \to B$ is $(n-1)$-connected. Since A_n is constructed from A_{n-1} by attaching n-cells, for every choice of basepoint we have $\pi_i(A_{n-1}) \approx \pi_i(A_n)$ for $i < n - 1$ and $\pi_{n-1}(A_n)$ is a quotient of $\pi_{n-1}(A_{n-1})$. For every map $\alpha\colon S^{n-1} \to A_{n-1}$ such that the composition with $A_{n-1} \to B$ is nullhomotopic, we've attached an n-cell, and so the composition $S^{n-1} \xrightarrow{\alpha} A_{n-1} \to$

A_n is nullhomotopic. Thus, $\pi_{n-1}(A_n) \to \pi_{n-1}(B)$ is an isomorphism for every choice of basepoint. In addition, for every map $\beta\colon D^n/S^{n-1} \to B$ for which the image of the collapsed S^{n-1} is in the image of $A_{n-1} \to B$, we've wedged on a copy of D^n/S^{n-1} to A_{n-1} and mapped it to B using β, and so $\pi_n(A_n) \to \pi_n(B)$ is surjective for every choice of basepoint. Thus, the map $A_n \to B$ is n-connected. This completes the induction. $\square$

We now let $\widetilde{B} = \operatorname{colim}_n A_n$. Since spheres and disks are compact, every map from a sphere or disk to $\operatorname{colim}_n A_n$ factors through some A_n, and so we have $\operatorname{colim}_n \pi_i A_n \approx \pi_i \widetilde{B}$ for $i \geq 0$. Since the map $\pi_i A_n \to \pi_i B$ is an isomorphism for $n > i$, the map $\pi_i \widetilde{B} \to \pi_i B$ is an isomorphism for $i \geq 0$, and so the map $\widetilde{B} \to B$ is a weak equivalence.

3.3. **The functoriality of the construction.** We will now show that the construction of Section 3.1 is functorial, i.e., that if we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

and we apply the construction of Section 3.1 to $A \to B$ to obtain $A \to \widetilde{B} \to B$ and to $A' \to B'$ to obtain $A' \to \widetilde{B}' \to B'$, then there is a natural commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ \widetilde{B} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{B}' \arrow[d] \\ B \arrow[r, "g"'] & B' \end{tikzcd} .$$

We define $\tilde{g}$ by defining $f_n\colon A_n \to A'_n$ inductively on the constructions of $\widetilde{B}$ and $\widetilde{B}'$.

To begin, we have

$$A_0 = A_{-1} \amalg \Big(\coprod_{D^0 \to B} D^0 \Big) \qquad \text{and} \qquad A'_0 = A'_{-1} \amalg \Big(\coprod_{D^0 \to B'} D^0 \Big)$$

and we define $f_0\colon A_0 \to A'_0$ by sending the copy of D^0 indexed by $\alpha\colon D^0 \to B$ to the copy of D^0 indexed by $g \circ \alpha\colon D^0 \to B'$.

For the inductive step, suppose that $n > 0$ and that we've defined $f_{n-1}\colon A_{n-1} \to A'_{n-1}$. The space A_n is constructed by attaching an n-cell to A_{n-1} for each commutative square

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & A_{n-1} \arrow[d] \\ D^n \arrow[r, "\beta"'] & B \end{tikzcd}$$

We take the cell attached to A_{n-1} by the map α to the cell attached to A'_{n-1} by the map $f_{n-1} \circ \alpha$ indexed by the outer commutative rectangle

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & A'_{n-1} \arrow[d] \\ D^n \arrow[r, "\beta"'] & B \arrow[r, "g"'] & B' \end{tikzcd}$$

Doing that for each n-cell attached to A_{n-1} defines $f_n\colon A_n \to A'_n$.

That completes the induction, and we let $\tilde{g}\colon \widetilde{B} \to \widetilde{B}'$ be $\operatorname{colim}_n f_n$.

4. The proof of Theorem 2.2

We construct the factorization in Section 4.1, show that the map $\widetilde{B} \to B$ is a weak equivalence in Section 4.2, and show that the construction is functorial in Section 4.3.

4.1. **The construction.** We use a modification of the construction of Section 3.1. We construct A_0 exactly as in Section 3.1, but when $n > 0$ and we are constructing A_n out of A_{n-1}, we attach only the n-cells indexed by commutative squares

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & A_{n-1} \arrow[d] \\ D^n \arrow[r, "\beta"'] & B \end{tikzcd}$$

for which $\alpha\colon S^{n-1} \to A_{n-1}$ is a cellular map.

4.2. **The homotopy groups of the spaces in the construction.**

Lemma 4.1. *For each $n \geq 0$ the map $A_n \to B$ is n-connected.*

Proof. We will show inductively on n that the map $A_n \to B$ is n-connected.

The space A_0 was constructed to map onto B, and so the map $A_0 \to B$ is 0-connected.

The space A_1 was constructed by attaching 1-cells to A_0 that connected any pair of vertices in A_0 whose images were in the same path component of B; since every path component of A_0 contains at least one vertex, the set of path components of A_1 maps isomorphically to the set of path components of B. In addition, a loop was wedged at every vertex of A_0 for every loop in B at the image of that vertex; since every path component of B contains the image of a vertex of A_0, for every basepoint of A_1 the fundamental group of A_1 maps epimorphically onto the fundamental group of B. Thus, the map $A_1 \to B$ is 1-connected.

Suppose now that $n > 1$ and that the map $A_{n-1} \to B$ is $(n-1)$-connected. Since A_n is constructed from A_{n-1} by attaching n-cells, for every choice of basepoint we have $\pi_i(A_{n-1}) \approx \pi_i(A_n)$ for $i < n-1$ and $\pi_{n-1}(A_n)$ is a quotient of $\pi_{n-1}(A_{n-1})$. For every cellular map $\alpha\colon S^{n-1} \to A_{n-1}$ such that the composition with $A_{n-1} \to B$ is nullhomotopic, we've attached an n-cell, and so the composition $S^{n-1} \xrightarrow{\alpha} A_{n-1} \to A_n$ is nullhomotopic. Since every map $S^{n-1} \to A_{n-1}$ is homotopic to a cellular map, $\pi_{n-1}(A_n) \to \pi_{n-1}(B)$ is an isomorphism for every choice of basepoint. In addition, for every map $\beta\colon D^n/S^{n-1} \to B$ for which the image of the collapsed S^{n-1} is in the

image of a vertex of A_{n-1}, we've wedged on a copy of D^n/S^{n-1} to that vertex of A_{n-1} and mapped it to B using β; since every path component of B is in the image of a vertex of A_{n-1}, $\pi_n(A_n) \to \pi_n(B)$ is surjective for every choice of basepoint. Thus, the map $A_n \to B$ is n-connected. This completes the induction. □

We now let $\widetilde{B} = \operatorname{colim}_n A_n$. Since spheres and disks are compact, every map from a sphere or disk to $\operatorname{colim}_n A_n$ factors through some A_n, and so we have $\operatorname{colim}_n \pi_i A_n \approx \pi_i \widetilde{B}$ for $i \geq 0$. Since the map $\pi_i A_n \to \pi_i B$ is an isomorphism for $n > i$, the map $\pi_i \widetilde{B} \to \pi_i B$ is an isomorphism for $i \geq 0$, and so the map $\widetilde{B} \to B$ is a weak equivalence.

4.3. **The functoriality of the construction.** We will now show that the construction of Section 4.1 is functorial, i.e., that if we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

in which $f\colon A \to A'$ is a cellular map and we apply the construction of Section 4.1 to $A \to B$ to obtain $A \to \widetilde{B} \to B$ and to $A' \to B'$ to obtain $A' \to \widetilde{B}' \to B'$, then there is a natural commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A' \arrow[d] \\ \widetilde{B} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{B}' \arrow[d] \\ B \arrow[r, "g"'] & B' \end{tikzcd} .$$

We define $\tilde{g}$ by defining $f_n\colon A_n \to A'_n$ inductively on the constructions of $\widetilde{B}$ and $\widetilde{B}'$. Since each $f_n\colon A_n \to A'_n$ is a cellular map, the composition of a cellular map $\alpha\colon S^{n-1} \to A'_n$ with $f_{n-1}\colon A_{n-1} \to A'_{n-1}$ is also cellular, and so we have an induced map $f_n\colon A_n \to A'_n$. Thus, the induction goes through, and we let $\tilde{g}\colon \widetilde{B} \to \widetilde{B}'$ be $\operatorname{colim}_n f_n$.

5. Proof of Proposition 2.3

Since we are factorizing the map $\emptyset \to B$, in the sequence $A_{-1} \to A_0 \to A_1 \to \cdots$ whose colimit is $\widetilde{B}$ (see Section 3.1) the space A_{-1} is empty. Thus, for each $n \geq 0$ the space A_n is an n-dimensional CW-complex, and so *every* map $S^n \to A_n$ is a cellular map. Thus, the sequence constructed in Section 4.1 is exactly the same as the sequence constructed in Section 3.1, and so their colimits are the same.

6. Proof of Theorem 2.4

We will show by induction that in the diagram

$$\begin{tikzcd}
\emptyset = A_{-1} \arrow[r] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \cdots \\
\emptyset = A'_{-1} \arrow[r] & A'_0 \arrow[r] & A'_1 \arrow[r] & A'_2 \arrow[r] & \cdots
\end{tikzcd}$$

used to construct $\widetilde{B} \to \widetilde{B}'$, each map $A_n \to A'_n$ is an inclusion of a subcomplex. The induction is begun because A_0 has one point for every point of B and A'_0 has one point for every point of B'.

Now assume that $n > 0$ and that $A_{n-1} \to A'_{n-1}$ is an inclusion of a subcomplex. Since the map $B \to B'$ is also an inclusion, the set of n-cells to be attached to A_{n-1} is a subset of the set of n-cells to be attached to A'_{n-1}, and so $A_n \to A'_n$ will also be an inclusion of a subcomplex.

7. The proof of Theorem 2.5

Let $X_S = \cap_{s\in S} X_s$.

- Let $\emptyset = A_{-1} \to A_0 \to A_1 \to \cdots$ be the sequence created in the proof of Theorem 2.1 whose colimit is $\mathrm{CW}(X)$,
- let $\emptyset = A^S_{-1} \to A^S_0 \to A^S_1 \to \cdots$ be the sequence created in the proof of Theorem 2.1 whose colimit is $\mathrm{CW}(X_S)$, and
- for each $s \in S$ let $\emptyset = A^s_{-1} \to A^s_0 \to A^s_1 \to \cdots$ be the sequence created in the proof of Theorem 2.1 whose colimit is $\mathrm{CW}(X_s)$.

The proof of Theorem 2.4 shows that A^S_n and A^s_n are subcomplexes of A_n for all $s \in S$ and $n \geq 0$; we will show by induction that $A^S_n = \cap_{s\in S} A^s_n$ for all $n \geq 0$.

Since A^S_0 is discrete with one point for each point of X_S and for all $s \in S$ the space A^s_0 is discrete with one point for each point of X_s, we have $A^S_0 = \cap_{s\in S} A^s_0$.

Assume now that $n > 0$ and $A^S_{n-1} = \cap_{s\in S} A^s_{n-1}$. The space A^S_n is constructed by attaching an n-cell to A^S_{n-1} for each commutative square

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & A^S_{n-1} = \cap_{s\in S} A^s_{n-1} \arrow[d] \\
D^n \arrow[r] & X_S = \cap_{s\in S} X_s
\end{tikzcd}$$

Since the maps $A^S_{n-1} \to A^s_{n-1}$ and $X_S \to X_s$ are inclusions for all $s \in S$, each such n-cell corresponds to a unique n-cell in $\cap_{s\in S} A^s_n$, i.e., the map $A^S_n \to \cap_{s\in S} A^s_n$ is an injection.

To see that the map $A^S_n \to \cap_{s\in S} A^s_n$ is a surjection, let

$$\left.\begin{tikzcd}
S^{n-1} \arrow[r, "\alpha_s"] \arrow[d] & A^s_{n-1} \arrow[d] \\
D^n \arrow[r, "\beta_s"'] & X_s
\end{tikzcd}\right\} \text{for every } s \in S$$

index n-cells of the A^s_n that together define an n-cell of $\cap_{s\in S} A^s_n$. Since the maps $A^s_{n-1} \to A_{n-1}$ and $X_s \to X$ are all inclusions, the compositions $S^{n-1} \xrightarrow{\alpha_s} A^s_{n-1} \to$

A_{n-1} are all equal and the compositions $D^n \xrightarrow{\beta_s} X_s \to X$ are all equal, and the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha_s"] \arrow[d] & A^s_{n-1} \arrow[r] & A_{n-1} \arrow[d] \\ D^n \arrow[r, "\beta_s"'] & X_s \arrow[r] & X \end{tikzcd}$$

(for any $s \in S$; the upper and lower compositions are all the same) indexes an n-cell that was attached to A_{n-1} when creating A_n. Since the upper composition factors uniquely through $\cap_{s\in S} A^s_{n-1}$ and the lower composition factors uniquely through $X_S = \cap_{s\in S} X_s$, those factorizations index an n-cell that was attached to A^S_{n-1} when creating A^S_n, and that n-cell maps to our n-cell of $\cap_{s\in S} A^s_n$.

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